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A note on a nonautonomous O.D.E. related to the Fisher equation

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Abstract

We give elementary proofs of the existence of a heteroclinic for the nonautonomous analogue of an O.D.E. arising in connection with the Fisher equation. In particular, we give a simple technique to compute lower bounds of the values of the admissible speeds. © 2000 Elsevier Science B.V. All rights reserved.

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1. Introduction

The purpose of this note is to study the existence of two classes of solutions, in unbounded intervals, to the nonlinear nonautonomous ordinary differential equation

$$u'' + cu' + a(t)f(u) = 0, \tag{1}$$

where $c > 0$, and the functions f and a are nonnegative. It is well known that, when $a(t) \equiv 1$ and f has a convenient form, (1) arises in connection with the Fisher equation that models a diffusion phenomenon in biomathematics (see [7,4]) and c represents the admissible speed of a one-dimensional travelling wave.

We are concerned with the situation where at least the following assumptions hold.

(H1) $f : \mathbb{R}_+ \to \mathbb{R}_+$ is a locally Lipschitz continuous function such that $f(0) = 0$, and there exists $d > 0$ such that $f(u) > 0$ if $0 < u < d$.

(H2) $a : \mathbb{R} \to \mathbb{R}$ is a continuous function such that, for some $\alpha > 0$, $\alpha < a(t) \leqslant 1$.

The solutions we look for are *positive*. Accordingly, the word *solution* will be used to mean *positive solution* throughout.

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We are interested in two kinds of problems:

I. In case where f has a zero, say $f(1)=0$, with $f(u)>0$ whenever $0<u<1$, we look for (monotonic) heteroclinics, that is, solutions $u(t)$ defined over $(-\infty,+\infty)$ with $u(-\infty)=1$ and $u(+\infty)=0$ and strictly decreasing.

II. We consider the existence of (nontrivial) positive solutions defined in an unbounded interval of the form $[t_0,+\infty)$ and satisfying $u(t_0)=0=u(+\infty)$.

Problem I (for the autonomous equation) has been widely studied, various approaches being available in the vast literature: we refer the reader to Aronson and Weinberger [4], Artstein and Slemrod [5], Kelley [6], Ahmad and Lazer [1] and the bibliography in those papers. Our purpose is to show that it is possible to study the nonautonomous case (1) as well using a very elementary technique based on comparison with linear equations. Some of our ideas have been inspired by the paper of Ahmad and Lazer [1]. We obtain, in particular, information on the range of values of c for which such a heteroclinic exists.

Problem II (which, by the way, is naturally suggested by phase plane analysis in the autonomous case) can be solved as a by-product of the same technique.

An important role is played by the function

$$g(u):=\frac{f(u)}{u},\quad u>0.$$

Study of the autonomous case has shown that a sufficient condition for the existence of the mentioned heteroclinic is a bound of the form $\sup_{0<u<1} g(u)=M<c^2/4$. This condition cannot be improved if $g(u)$ is decreasing in $(0,1)$, as in the classical model $f(u)=u(1-u)$ (cf. [4]). In cases where $g(0)=0$ (which apply for instance to the model $f(u)=u^n(1-u)$, $n>1$) this range of admissible values of c may be slightly extended; we shall show in Section 3 how to systematically compute a better (although not optimal) lower bound.

In the next section we collect a series of lemmas that will be used in the sequel. In Sections 3 and 4 we deal, respectively, with problems I and II.

2. Auxiliary results

We start with three simple but useful observations that are consequences of the positivity of c, a and f.

Remarks. (1) The solution for the initial value problem for (1) with $u(t_0)=u_0\geqslant 0$ and $u'(t_0)=u_1\geqslant 0$ exists in the maximal interval $[t_0,s)$ with $s<+\infty$ only if $\lim_{t\to s} u(t)=0$. (2) A nonconstant solution of (1) has at most a critical point, which must be an absolute maximum. (3) If $u(t)$ is a solution of (1) with $0\leqslant u(t)\leqslant p$ $\forall t\geqslant t_0$ and $\sup_{0\leqslant u\leqslant p} f(u)=N$ then $\inf_{t\geqslant t_0} u'(t)\geqslant\min\{u'(t_0),-N/c\}$.

Lemma A. *Let $c>0$, $p>0$, $c^2\geqslant 4M$ and $0\leqslant\varepsilon\leqslant p(c/2+\sqrt{c^2-4M}/2)$. Then the solution $u(t)$ of the initial value problem*

$$u''+cu'+Mu=0, \tag{2}$$

$$u(0)=p,\quad u'(0)=-\varepsilon \tag{3}$$

is positive in $[0,+\infty)$ and tends to zero as $t\to+\infty$.

Proof. Assume $c^2 > 4M$. The final assertion is obvious since, with $s = (c/2 + \sqrt{c^2 - 4M}/2)$ and $r = (c/2 - \sqrt{c^2 - 4M}/2)$

$$u(t) = \frac{\varepsilon - pr}{s - r}\exp(-st) + \frac{ps - \varepsilon}{s - r}\exp(-rt)$$

and the positiveness follows from the fact that $u'(t)$ vanishes exactly once, certainly not for $t > 0$ provided $(prs - \varepsilon s)/(prs - \varepsilon r) \leqslant 1$. The proof is equally simple in case $c^2 = 4M$. □

Lemma B. *Let $l(t)$, $m(t)$ be continuous functions in $\mathbb{R}$ such that $l(t) \leqslant m(t)\ \forall t \in \mathbb{R}$ and $u(t)$, $v(t)$ be the respective solutions of*

$$u'' + cu' + l(t)u = 0, \quad v'' + cv' + m(t)v = 0$$

with $u(t_0) = v(t_0) \geqslant 0$ and $u'(t_0) = v'(t_0)$. Then, provided $v(t) > 0$ in $(t_0, t_1]$ it follows that $u(t) \geqslant v(t)$ in $(t_0, t_1]$ and $u(t_1) = v(t_1)$ only if $l(t) \equiv m(t)$ in $[t_0, t_1]$. Further, if $v'(t_1) = 0$, then $u'(t_1) \geqslant 0$.

Proof. Multiplying the equations by v and u, respectively, integrating and subtracting we obtain

$$[\exp(ct)(u'(t)v(t) - u(t)v'(t)]_{t_0}^{t_1} + \int_{t_0}^{t_1} \exp(ct)u(t)v(t)(l(t) - m(t))\,\mathrm{d}t = 0. \tag{4}$$

Starting with $u'(t_0) = v'(t_0) + \delta$, $\delta > 0$, we see that $u(t) > v(t)$ in (t_0, t_1) and $u(t_1) = v(t_1)$ leads to a contradiction. The first conclusion follows passing to the limit as $\delta \to 0$. The second conclusion follows from (4) as well. □

The proofs of the following two lemmas are trivial exercises:

Lemma C. *The first eigenvalue $\lambda = \lambda(b)$ of the problem*

$$u'' + cu' + \lambda u = 0, \quad u(0) = 0,\ u'(b) = 0 \tag{5}$$

is $\lambda = (c^2/4) + (\alpha^2/4)$, *where α is the smallest positive root of* $\tan(b\alpha/2) = \alpha/c$ *if* $0 < b < 2/c$, $\lambda = c^2/4$ *if $b = 2/c$ and $\lambda = c^2/4 - \alpha^2/4$, where α is the positive root of* $\exp(b\alpha) = (c + \alpha)/(c - \alpha)$, *if $b > 2/c$.*

Lemma D. *Suppose that $0 \leqslant f(u) \leqslant b$ if $0 \leqslant u \leqslant p$. Then if $A > (c + 1)p + b$ there exists $\bar{t} \leqslant 1$ such that the solution $u(t)$ of*

$$u'' + cu' + a(t)f(u) = 0, \tag{1}$$

$$u(0) = 0, \quad u'(0) = A \tag{6}$$

satisfies $u(\bar{t}) = p$ and $u'(\bar{t}) \geqslant A - cp - b$.

Lemma E. *Assume that $f(u) > 0$ if $u > 0$. Then for every $A > 0$ the solution of* (1)–(6) *has a critical point.*

Proof. Fix $p=u(\bar{t})$, where $\bar{t}>0$ and $u'(\bar{t})=m>0$. Let $0<\delta<\min\{c^2/4, \inf_{[p,p+m/c]} g(u)\}$. Compare $u(t)$ and the solution of

$$v''+cv'+(\alpha\delta)v=0, \quad v(\bar{t})=p, \;\; v'(\bar{t})=m.$$

Using Lemma B we see that this in turn lies below the solution $z(t)$ of

$$z''+cz'=0, \quad z(\bar{t})=p, \;\; z'(\bar{t})=m$$

and therefore $v(t)<p+m/c$. It follows, using Lemma B and our choice of δ, that $u\leqslant v$ as long as u remains above p. Since $v=[(m+ps)/(s-r)]\exp(-r(t-\bar{t}))-[(m+pr)/(s-r)]\exp(-s(t-\bar{t}))$ for some $s>r>0$, u has a (unique) critical point b. □

Remark. Assume, in addition, that $\liminf_{u\to+\infty} g(u)>0$. Then in the above argument we can fix $p>0$ and then δ independently of (large) m. Thus $u'(b)=0$ implies $b\leqslant t_1$ where t_1 is the critical point of v (see Lemma B). It is easy to see that $t_1\to T_1:=\bar{t}+(\ln s-\ln r)/(s-r)$ as $m\to+\infty$. It follows that the critical point b of u is bounded above, independently of (large) m.

Lemma F. *Assume* (H1). *Then provided that A is sufficiently small, the solution $u(t)$ of* (1)–(6) *has a critical point b and its corresponding value $u(b)$ can be made arbitrarily small.*

Proof. The argument of the preceding proof shows that, for small m, u must have a critical point (recall that the maximum of v is less than $p+m/c$). Choose p arbitrarily small. If A is sufficiently small and the maximum of u is not less than p then $u(\bar{t})=p$ for some (large) $\bar{t}$ and $u'(\bar{t})=m<A$. Again by the above proof, $u<v$ for $t>\bar{t}$ and the conclusion is obvious. □

3. Heteroclinics

In this section we introduce the new assumption

(H3) $f(1)=0$ and $f(u)>0$ if $0<u<1$.

Proposition 3.1. *Assume* (H1)–(H2). *Let $p\leqslant d$, $u(t)$ be a solution of* (1) *with $u(t_0)=p>0$, let $M:=\sup_{0<u<p} g(u)$ and assume $c^2\geqslant 4M$ and $u'(t_0)=-\varepsilon$ with $0<\varepsilon\leqslant p(c/2+\sqrt{c^2-4M}/2)$. Then $u(t)$ is defined, decreasing and positive in $[t_0,+\infty)$ and $u(+\infty)=0$.*

Proof. We compare $u(t)$ with the solution of

$$v''+cv'+Mv=0,$$

$$v(t_0)=p, \quad v'(t_0)=-\varepsilon.$$

By Lemma A $v(t)$ is positive in $[t_0,+\infty)$ and by Lemma B and the definition of M we have $u(t)>v(t)$ in $[t_0,+\infty)$. Hence $u(t)$ is positive, decreasing and $l:=\lim_{t\to+\infty} u(t)$ exists. From

$$u'(t)+\varepsilon+c(u(t)-p)+\int_{t_0}^{t} a(s)f(u(s))\,\mathrm{d}s=0, \quad \forall t\geqslant t_0$$

using the fact that $a(s) \geqslant \alpha > 0$, the boundedness of $u'(t)$ and positiveness of f, one deduces that $\int_{t_0}^{+\infty} f(u(s))\,\mathrm{d}s$ is finite. Hence $l = 0$. □

Proposition 3.2. *Assume* (H1)–(H3). *Assume that there exists* $p \in (0,1)$ *such that* $N := \sup_{0<u<1} f(u)$ *and* $M := \sup_{0<u<p} g(u)$ *satisfy* $c^2 > 4M$ *and*

$$\frac{N}{c} \leqslant p\left(\frac{c}{2} + \frac{\sqrt{c^2 - 4M}}{2}\right). \tag{7}$$

Then for each sufficiently small $\varepsilon > 0$ *the solution* $u(t, t_0, \varepsilon)$ *of* (1) *such that* $u(t_0, t_0, \varepsilon) = 1$ *and* $u'(t_0, t_0, \varepsilon) = -\varepsilon$ *is positive in* $[t_0, +\infty)$ *and*

$$\lim_{t \to +\infty} u(t, t_0, \varepsilon) = 0.$$

Proof. The solution $u(t, t_0, \varepsilon)$ has no critical points and therefore is strictly decreasing. It cannot remain above a positive constant by the argument used in the proof of Proposition 3.1. Let $\bar{t}$ be such that $u(\bar{t}, t_0, \varepsilon) = p$. The equation itself shows that $u'(\bar{t}, t_0, \varepsilon) \geqslant -N/c$ (consider separately the cases where $\bar{t}$ lies in an interval of convexity or of concavity of the solution) and therefore Proposition 3.1 can be applied. □

Remark. Note that this result involves only the bounds α and 1 of function a and not the behaviour of a itself.

Theorem 3.3. *Let all the hypotheses of Proposition* 3.2 *hold. Then* (1) *has a decreasing heteroclinic solution connecting* 1 *and* 0.

Proof. Take a sequence t_n decreasing to $-\infty$ and consider the solution $u(., t_1, \varepsilon_1)$ where ε_1 is a small positive number. According to Proposition 3.2, $0 < u(t, t_1, \varepsilon_1) \leqslant 1$ for $t \geqslant t_1$ and there exists $\bar{t}$ such that $u(\bar{t}, t_1, \varepsilon_1) = \frac{1}{2}$.

Claim. There exists $n_2 > 1$ such that

$$u(\bar{t}, t_{n_2}, \varepsilon_1) < \tfrac{1}{2}.$$

Proof. Otherwise we would have $u(\bar{t}, t_n, \varepsilon_1) \geqslant \frac{1}{2}$ $\forall n > 1$ and, by a t_n translation, this can be written

$$u_n(\bar{t} - t_n) \geqslant \tfrac{1}{2} \tag{8}$$

in terms of the solution of

$$u_n'' + cu_n' + a_n(t) f(u_n) = 0, \quad u_n(0) = 1, \;\; u_n'(0) = -\varepsilon_1$$

(where $a_n(t) = a(t + t_n)$). The boundedness of a_n, u_n and u_n' and Ascoli's theorem enable us, by extracting subsequences and a diagonal procedure, to suppose that

$$a_n \to a_\infty \quad \text{in } L^\infty\text{weak-}^*, \;\; \alpha \leqslant a_\infty(t) \leqslant 1,$$

$$u_n \to u \text{ in } C^1(K), \; K \text{ any compact interval in } [0, +\infty).$$

Since

$$u'' + cu' + a_\infty(t)f(u) = 0, \quad u(0) = 1, \; u'(0) = -\varepsilon_1$$

(and it is easily seen that Proposition 3.2 still applies to solutions in the Carathéodory sense) there exists $\tilde{t}$ such that $u(\tilde{t}) = \frac{1}{4}$. Since $u_n \to u$ uniformly in $[0, \tilde{t}+1)$ and $\bar{t} - t_n \to +\infty$ this contradicts (8) and so the Claim holds. □

To go on with the proof we observe that if $\delta > 0$ is sufficiently small we have $u(\bar{t}, t_{n_2}, \delta) > \frac{1}{2}$, since $u(., t_{n_2}, \delta) \to 1$ as $\delta \to 0^+$ in $[t_{n_2}, \bar{t}]$. By the intermediate value theorem we can pick up $0 < \varepsilon_2 < \varepsilon_1$ such that $u(\bar{t}, t_{n_2}, \varepsilon_2) = \frac{1}{2}$. This argument can be iterated so as to construct decreasing sequences $\tau_k = t_{n_k}$ and ε_k with the property that $u(\bar{t}, \tau_k, \varepsilon_k) = \frac{1}{2}$.

Using again the boundedness of $u(., \tau_k, \varepsilon_k)$ and $u'(., \tau_k, \varepsilon_k)$ and the diagonal procedure we can pass to a subsequence (which for convenience is denoted by the same symbol) so that for any compact interval $K \subset \mathbb{R}$

$$u(., \tau_k, \varepsilon_k) \to u \quad \text{in } C^1(K).$$

The limit function u thus obtained is, of course, a decreasing solution to (1), such that $u(\bar{t}) = \frac{1}{2}$ and $0 < u(t) < 1 \;\forall t \in \mathbb{R}$ (by the uniqueness theorem for the initial value problem u cannot take the values 0 or 1). Finally, we can use the argument of the proof of Proposition 3.1 again to conclude that $\lim_{t\to-\infty} u(t) = 1$, $\lim_{t\to+\infty} u(t) = 0$ and $\lim_{t\to\pm\infty} u'(t) = 0$. □

Remark. It can be shown, using an argument similar to the proof of the Claim, that $\varepsilon_k \to 0$.

This theorem shows the significance of the range of values of c such that (7) holds. Let us work out the example $f(u) = u^n(1-u)$, $n > 1$. Then $N = n^n/(n+1)^{n+1}$ and $g(u)$ is increasing in $[0, (n-1)/n]$. We therefore look for the points $(p, c) \in (0, (n-1)/n) \times (0, +\infty)$ such that

$$\frac{n^n}{(n+1)^{n+1}} = p\left(\frac{c^2}{2} + c\frac{\sqrt{c^2 - 4p^{n-1}(1-p)}}{2}\right). \tag{7'}$$

In fact we must restrict p to the interval $(0, \bar{p})$ where $n^n/2(n+1)^{n+1} = f(\bar{p})$, since in this interval (7′) does define a continuous function $c = c(p)$. The minimum attained by $c(p)$ yields, of course, a lower bound for the values of c such that (1) with our choice of f has a heteroclinic. For instance, if $n = 3$ we have $\bar{p} \approx 0.4607$, and we obtain for the minimum $c^\star$ the approximated value $c^\star \approx 0.6392$ so that the heteroclinic connecting 1 and 0 exists at least for $c > c^\star$. Note that the lower bound given by the less accurate formula coming from Proposition 3.1 ($2\sqrt{\sup_{0<u<1} g(u)}$) is approximately 0.7698.

4. Positive solutions in an unbounded interval

In this section we consider the problem of finding (positive) solutions to

$$u'' + cu' + a(t)f(u) = 0, \tag{1}$$

$$u(0) = 0 = u(+\infty) \tag{9}$$

(for definiteness the initial endpoint is taken to be $t_0 = 0$).

Proposition 4.1. *Let $f(u)>0$ $\forall u>0$. Given a positive number u_0 let $N=\sup_{0<u<u_0} f(u)$. Suppose that for some $p\in(0,u_0)$ $M=\sup_{0<u<p} g(u)<c^2/4$ satisfies $N/c<p(c/2+\sqrt{c^2-4M}/2)$. Then there exists a positive solution of* (1)–(9) *whose maximum is u_0.*

Proof. Consider the initial value problem (1) with $u(0)=0$ and $u'(0)=d>0$. We have seen (Lemmas E and F) that its solution $u(t,d)$ has a maximum value $m(d)$ and that $m(+\infty)=+\infty$, $m(0^+)=0$. In particular, u_0 is the maximum of some solution, say $u_0=u(t_0,d)$. Continuing this solution to the interval $(t_0,+\infty)$ (with initial value p and initial slope zero), the argument of Proposition 3.2 allows us to conclude. □

Corollary 4.2. *Let $f(u)>0$ $\forall u>0$. Given a positive number u_0 such that $M=\sup_{0<u<u_0} g(u)<c^2/4$ then for every $q\in(0,u_0)$ there exists a positive solution of* (1)–(9) *whose maximum is q.*

Remark. One can formulate several conditions that guarantee the existence of one solution $u(t)$ of (1) such that, for some $b>0$,

$$u(0)=0,\quad u'(b)=0. \tag{10}$$

The fact that the continuation of this solution to the interval $[b,+\infty)$ defines a solution of (1)–(9) depends on the magnitude of $u_0=u(b)$ and the verification of the hypothesis and the assumption on M of Proposition 4.1 or Corollary 4.2. Let us give two examples, where we assume not only (H1)–(H3) but also $\sup_{0<u<1} g(u)\leqslant c^2/4$.

1. Suppose that for some $\varepsilon>0$ and $\delta>0$ we have $g(u)\geqslant\delta$ whenever $0\leqslant u\leqslant\varepsilon$. Then there exists b_0 such that for $b\geqslant b_0$ some small multiple of the first eigenfunction of (5) is a lower solution of (1)–(10). Since 1 is an upper solution for the same problem, there exists a nontrivial solution of (1)–(9) whose maximum is attained at b. The value of b_0 can be estimated using Lemma C.
2. First we remark that, extending f to $\mathbb{R}\setminus[0,1]$ with the value 0, a solution of problem (1)–(10) thus extended is in fact a solution of our original problem. Hence, we can assume that we are dealing with that extension.

Let $F(u):=\int_0^u f(s)\,\mathrm{d}s$. Assume that

$$m:=\limsup_{u\to0^+}\frac{2F(u)}{u^2}<\frac{c^2}{4}.$$

Solutions of (1)–(10) are precisely the critical points of the functional

$$J_b(u)=\int_0^b \mathrm{e}^{ct}\left(\frac{u'^2}{2}-a(t)F(u)\right)\mathrm{d}t$$

defined in $H_b:=\{u\in H^1(0,b):u(0)=0\}$. Now, for any b such that $m<\lambda(b)<c^2/4$ (in particular $b>2/c$, according to Lemma C), J_b attains a strict local minimum at the origin, $J_b(0)=0$. On the other hand, consider the function $\phi\in H_b$ defined as $\phi(t)=(ce/2)t\mathrm{e}^{-(c/2)t}$ if $0\leqslant t\leqslant 2/c$, $\phi(t)=1$ if $2/c\leqslant t\leqslant b$. (Note that the restriction of ϕ to $[0,2/c]$ is a first eigenfunction corresponding to the eigenvalue $c^2/4$.) An easy calculation gives the estimate

$$J_b(\phi)<\frac{c\mathrm{e}^2}{12}-\alpha\int_0^{2/c}\exp(ct)F\left(\frac{ce}{2}t\mathrm{e}^{-(c/2)t}\right)\mathrm{d}t-\frac{\alpha(\exp(cb)-\mathrm{e}^2)F(1)}{c}.$$

It follows that, at least if we can take b large enough (which is allowed if m is sufficiently small) $J_b(\phi) < 0$. Since J_b is a coercive functional, we obtain, for all such values of b, two additional critical points of J_b (see [8] for details), that is, two nontrivial solutions of (1)–(10), and therefore of (1)–(9) as well.

Finally, let us consider the case where $f(u) > 0$ for $u > 0$, $\sup_{u>0} g(u) \leqslant c^2/4$ and in addition

$$\limsup_{u\to 0^+} \frac{2F(u)}{u^2} < \alpha \liminf_{u\to+\infty} g(u). \tag{11}$$

Take any b such that

$$\limsup_{u\to 0^+} \frac{2F(u)}{u^2} < \lambda = \lambda(b) < \alpha \liminf_{u\to+\infty} g(u). \tag{12}$$

As we have seen, $J_b(u)$ has a local minimum at the origin in H_b. On the other hand, using arguments of Amann [2], we can prove that there is a bound for the L^∞-norm of (positive) solutions of

$$u'' + cu' + a(t)f(u) + p(t) = 0, \quad u(0) = 0, \ u'(b) = 0, \tag{13}$$

where p is a positive continuous function, and that bound is independent of p:

Let ϕ be a positive first eigenfunction of (5). Multiplying (13) by $\phi(t)$ and integrating, using the right-hand side of (12), we easily find out that $\int_0^b (\mathrm{e}^{ct}u')'\phi(t)\,\mathrm{d}t$ is bounded independently of p. From this and the behaviour of ϕ it follows easily that there exists $M > 0$ such that

$$|u|_\infty < M$$

for all positive functions p and all solutions of (13).

Now fix $p > 0$. The gradient of J_b, viewed as an operator in H_b, can be written $J_b'(u) = u - K(N(u))$ where K is the compact linear operator that maps p into u via (13) with the term $a(t)f(u)$ suppressed and N is the Niemytski operator associated to $a(\cdot)f(u(\cdot))$. Hence problem (1)–(10) can be written in the form of a fixed point equation $u = K(N(u))$, $u \in H_b$. Consider the homotopy

$$u = K(N(u)) + \mu K(p), \quad 0 \leqslant \mu \leqslant \mu_0 \tag{14}$$

so that $\mu_0|K(p)|_\infty > M$. As K leaves the cone of positive functions invariant, (14) has no solution at all for $\mu = \mu_0$. Using $\deg(T, B_r)$ to denote the Leray–Schauder degree of the operator T with respect to 0 in the open ball centered at the origin and with radius r, it follows that $\deg(\mathrm{id} - KN, B_M) = 0$. A theorem of Amann [3] implies that, for sufficiently small ε, $\deg(\mathrm{id} - KN, B_\varepsilon) = 1$. The excision property of the fixed point index allows to conclude the existence of a solution of (1)–(10) in $B_M \setminus B_\varepsilon$. Once more we conclude that there exists a (nontrivial) solution of (1)–(9) attaining its maximum at b.

References

[1] S. Ahmad, A. Lazer, An elementary approach to travelling front solutions to a system of N competition-diffusion equations, Nonlinear Anal. TMA 16 (1991) 893–901.

[2] H. Amann, Fixed point equations and nonlinear eigenvalue problems in ordered Banach spaces, SIAM Rev. 18 (1976) 620–709.

[3] H. Amann, A note on degree theory for gradient mappings, Proc. AMS 84 (1982) 591–595.

[4] D.G. Aronson, H.F. Weinberger, Multidimensional nonlinear diffusion arising in population genetics, Adv. Math. 30 (1978) 33–76.
[5] Z. Artstein, M. Slemrod, Trajectories joining critical points, J. Differential Equations 44 (1982) 40–62.
[6] W. Kelley, Analytic approximations of travelling wave solutions, Applicable Anal. 55 (1994) 287–299.
[7] A. Kolmogoroff, I. Petrovskii, N. Piskunov, Étude de l'équation de la diffusion avec croissance de la quantité de matière et son application à un problème biologique, Bull. Univ. Moscow, Ser. Internat. Sec. A 17 (1937) 1–25.
[8] P.H. Rabinowitz, Minimax methods in critical point theory and applications to differential equations, CBMS Reg. Conf. 65, Amer. Math. Soc., Providence, RI, 1986.